General quantum Brownian motion with initially correlated and nonlinearly coupled environment

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The dynamics of an open quantum system exhibiting the quantum Brownian motion is analyzed when the coupling between the system and its environment is nonlinear, and the system and the reservoir are *initially correlated*. For couplings quadratic in the environment variables, the influence functional for the system is obtained perturbatively up to second order in the coupling constant, and then the propagator is explicitly evaluated when the particle is under the influence of a harmonic potential and an additional anharmonic potential, the so-called washboard potential. As an application of the propagator, the master equation and the Wigner equation are obtained for the quantum Brownian particle moving in a harmonic potential for the generalized correlated initial condition, and then for the specific case of the simplified "thermal" initial condition. The system is shown to obey the corresponding fluctuation-dissipation theorem.

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I. INTRODUCTION

The fluctuating or "Brownian" motion of a quantum particle coupled to an environment serves as a model for the investigation of observable macroscopic effects in open quantum systems. The problem is usually handled using the influence functional method introduced by Feynman and Vernon [1], where the object of interest is the reduced dynamics of the system evolving under the influence of the environment, which is quantified by the influence functional [2]. In the model studied by Caldeira and Leggett [3], the coordinate of the particle was coupled *linearly* to an infinite set of harmonic oscillators constituting the environment, and it was also assumed that the system and the environment were initially factorized. However, in general, there are physical problems modeled by the quantum Brownian motion, which involve some form of nonlinearity in the interaction between the system and its environment, for example, as pointed out by Hu, Paz, and Zhang [4], in the strong-field conditions in the early Universe when one cannot exercise any control over the strength of the coupling. Hu *et al.* [4] and Brun [5] developed techniques to obtain the influence functional perturbatively for the case of nonlinear system-environment coupling, but with the assumption of factorized initial conditions.

There may be difficulties associated with the factorized (product) initial state, which assumes a sudden artificial switch-on of the interaction between the system and the environment at time $t=0$, and thus influences the subsequent short-time behavior of the system. The treatment of the quantum Brownian motion with linear coupling has been generalized to the physically reasonable initial condition of a mixed state of the system and its environment by Hakim and Ambegaokar [6], Smith and Caldeira [7], and Grabert, Schramm, and Ingold [8], and by us [9] for the case of a system in a Stern-Gerlach potential. There is no treatment available in the literature for the quantum Brownian motion with nonlinear system-environment couplings and generalized nonfactorizable initial conditions, which is the aim of the present work.

In this work we use the influence functional to get the propagator for the problem when the particle is under the influence of a harmonic potential, and also an anharmonic potential, the so-called washboard potential [10,11], which models the motion of a heavy charged particle in the interior or at the surface of a metal. From the propagator for the particle in a harmonic potential we obtain the master equation and the Wigner equation for a general nonfactorizable initial condition, and then for the specific case of a "thermal" initial condition, introduced by Hakim and Ambegaokar [6], where the off-diagonal elements of the density matrix of the total system are suppressed and thus the transients due to the switching on of the system-environment interaction at $t=0$ are avoided. We also establish a fluctuation-dissipation theorem connecting the dissipative and the fluctuating influences of the stochastic environment.

The present paper is organized as follows. In Sec. II we briefly present the influence functional for nonlinear system-environment couplings [12]. In particular, for interactions quadratic in the position of the environmental oscillators coupled with an arbitrary function $f(x)$ of the system, the influence functional is obtained up to second order in the coupling constant. In Sec. III we use the influence functional with the simple form of coupling linear in the system coordinate, i.e., with $f(x)=x$, to obtain the propagator of the particle in a harmonic potential (Sec. III A), and in an additional anharmonic "washboard" potential (Sec. III B). In Sec. IV A we use the propagator in Sec. III A to obtain the master equation and from it the Wigner equation for the quantum Brownian oscillator. The inhomogeneities in the master equation under generalized initial conditions disappear for the case of the "thermal" initial conditions considered in Sec. IV B. In Sec. V we establish a generalized

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fluctuation-dissipation theorem for our quantum Brownian oscillator. Finally in Sec. VI we summarize our results.

II. INFLUENCE FUNCTIONAL

Our model for the quantum Brownian motion consists of the usual system of a quantum particle of mass M moving in a potential $V(x)$ and coupled to an environment of a set of harmonic oscillators. The actions for the system and the environment are given by

$$S_S[x]=\int_0^t ds\left[\frac{1}{2}M\dot{x}^2-V(x)\right] \tag{1}$$

and

$$S_E[\{q_n\}]=\int_0^t ds\sum_{n=1}^{N}\left[\frac{1}{2}m_n\dot{q}_n^2-\frac{1}{2}m_n\omega_n^2q_n^2\right], \tag{2}$$

where subscripts S and E stand for system (particle) and environment, respectively; x denotes the position of the particle; m_n, ω_n, and q_n are the mass, frequency, and position of the nth environmental oscillator. The action for the system-environment interaction is [4]

$$S_{SE}[x,\{q_n\}]=\int_0^t ds\sum_n\,[-\lambda C_n f(x)q_n^k], \tag{3}$$

where λ is a dimensionless coupling constant introduced for later use as a small parameter in the perturbative expansion, k is an integer, the system-environment coupling is nonlinear with the environmental nonlinearity resting in the power k of q_n and the system effect being described by a general function $f(x)$.

The influence functional produced by the environment is

$$\begin{aligned}\tilde{F}[x,x',\overline{x}]=\prod_n\int dq_{n_i}dq'_{n_i}dq_{n_f}Z_{R_n}^{-1}\int Dq_nDq'_nD\overline{q_n}\\ \times\exp\bigg[\frac{i}{\hbar}(S_E[\{q_n\}]+S_{SE}[x,\{q_n\}]-S_E[\{q'_n\}]\\ -S_{SE}[x',\{q'_n\}])-\frac{1}{\hbar}(S_E^{EC}[\{\overline{q_n}\}]\\ +S_{SE}^{EC}[\overline{x},\{\overline{q_n}\}])\bigg],\end{aligned} \tag{4}$$

assuming that initially the interacting system is in a thermal equilibrium state at a temperature $T=(k_B\beta)^{-1}$, with

$$\begin{aligned}q_n(t)=q'_n(t)=q_{n_f},\quad q_n(0)=\overline{q_n}(\hbar\beta)=q_{n_i},\\ \overline{q_n}(0)=q'_n(0)=q'_{n_i},\quad \overline{x}(0)=\overline{x'},\quad \overline{x}(\hbar\beta)=\overline{x}.\end{aligned} \tag{5}$$

In Eq. (4), Z_R^{-1} is a normalization constant that ensures that $\tilde{F}=1$ in the case of vanishing interactions. Superscript EC in Eq. (4) stands for the Euclidean action, and the corresponding imaginary-time functional integral over all paths $\overline{q_n}$ connecting $\overline{q_n}(0)=q'_{n_i}$ with $\overline{q_n}(\hbar\beta)=q_{n_i}$ describes initial correlations between the system and its environment [8]. The propagator in the path-integral representation is given by

$$\begin{aligned}J(x_f,x'_f,t,x_i,x'_i,\overline{x},\overline{x'})=\frac{1}{Z}\int DxDx'D\overline{x}\exp\bigg[\frac{i}{\hbar}(S_S[x]\\ -S_S[x']+iS_S^{EC}[\overline{x}])\bigg]\tilde{F}[x,x',\overline{x}],\end{aligned} \tag{6}$$

where $S_S[x]$ is action (1) describing the system and $S_S^{EC}[\overline{x}]$ is its Euclidean counterpart.

Now, the influence functional is

$$\begin{aligned}\tilde{F}[x,x',\overline{x}]&=\prod_n\tilde{F}_n[x,x',\overline{x}]\\ &=\prod_n\exp\left[\frac{i}{\hbar}\delta A_n[x,x',\overline{x}]\right]\\ &=\exp\left[\frac{i}{\hbar}\delta A[x,x',\overline{x}]\right],\end{aligned} \tag{7}$$

where $\delta A=\Sigma_n\delta A_n$ is the total influence action.

As a consequence of the nonlinearity in the coupling, the influence functional cannot be obtained exactly. However, assuming that the parameter λ is small, we proceed as do Hu *et al.* [4] to obtain the desired influence functional perturbatively in orders of λ.

The influence functional of an environment where the environment oscillators are linearly coupled to the coordinate of the system is [4]

$$\begin{aligned}\tilde{F}_n^{(1)}[J,J',\overline{J}]\\ =\int_{-\infty}^{\infty}dq_{n_i}dq'_{n_i}dq_{n_f}Z_{R_n}^{-1}\int_{q_{n_i}}^{q_{n_f}}Dq_n\int_{q'_{n_i}}^{q_{n_f}}Dq'_n\int_{q'_{n_i}}^{q_{n_i}}D\overline{q_n}\\ \times\exp\bigg[\frac{i}{\hbar}\bigg\{S_E[q_n]+\int_0^t dsJ(s)q_n(s)-S_E[q'_n]\\ -\int_0^t dsJ'(s)q'_n(s)\bigg\}-\frac{1}{\hbar}\bigg\{S_E^{EC}[\overline{q_n}]\\ +\int_0^{\hbar\beta}d\tau[-\overline{J}(\tau)\overline{q_n}(\tau)]\bigg\}\bigg]\\ =\bigg\langle\exp\bigg[\frac{i}{\hbar}\bigg\{\int_0^t dsJ(s)q_n(s)-\int_0^t dsJ'(s)q'_n(s)\\ -i\int_0^{\hbar\beta}d\tau\overline{J}(\tau)\overline{q_n}(\tau)\bigg\}\bigg]\bigg\rangle_0,\end{aligned} \tag{8}$$

where $\langle\cdots\rangle_0$ denotes the average of the function of the environment variables. Using this definition, the influence functional can be expressed as [12]

$$\tilde{F}_n[x,x',\bar{x}] = \left\langle \exp\left[\frac{i}{\hbar}\{S_{SE}[x,q_n]-S_{SE}[x',q_n']+iS_{SE}^{EC}[\bar{x},\bar{q}_n]\}\right]\right\rangle_0$$

$$= \exp\left[\frac{i}{\hbar}\left\{S_{SE}\left[x,\frac{\hbar}{i}\frac{\delta}{\delta J}\right]-S_{SE}\left[x',-\frac{\hbar}{i}\frac{\delta}{\delta J'}\right]+iS_{SE}^{EC}\left[\bar{x},\hbar\frac{\delta}{\delta\bar{J}}\right]\right\}\right]\tilde{F}_n^{(1)}[J,J',\bar{J}]|_{\{J,J',\bar{J}=0\}}. \tag{9}$$

We now consider the case where the coupling is quadratic in the environment variables, i.e., where $k=2$ in Eq. (3). The influence functional up to second order in λ is then obtained in the continuum limit as

$$\tilde{F}[x,x',\bar{x}] = \exp\left\{-\frac{i}{\hbar}\int_0^t ds\,\delta V(x)+\frac{i}{\hbar}\int_0^t ds\,\delta V(x')-\frac{1}{\hbar}\int_0^{\hbar\beta}d\tau\,\delta\bar{V}(\bar{x})+\frac{1}{2\hbar}\int_0^{\hbar\beta}d\tau\int_0^{\hbar\beta}d\sigma\,k^{(2)}(\tau-\sigma)f(\bar{x}(\tau))f(\bar{x}(\sigma))\right.$$
$$+\frac{i}{\hbar}\int_0^t ds\int_0^{\hbar\beta}d\tau K^{*(2)}(s-i\tau)f(\bar{x}(\tau))[f(x(s))-f(x'(s))]-\frac{i}{\hbar}\int_0^t ds\int_0^s du[f(x(s))-f(x'(s))]\eta^{(2)}(s-u)$$
$$\left.\times[f(x(u))+f(x'(u))]-\frac{1}{\hbar}\int_0^t ds\int_0^s du[f(x(s))-f(x'(s))]\nu^{(2)}(s-u)[f(x(u))-f(x'(u))]\right\}. \tag{10}$$

In Eq. (10), $\delta V(x)$, $\delta V(x')$, $\delta\bar{V}(\bar{x})$ are the potential shifts, given by

$$\delta V(x) = \sum_n \delta V_n(x) = \sum_n \hbar\frac{\lambda C_n}{2m_n\omega_n}\mathcal{Z}f(x) = \int_0^\infty\frac{d\omega}{\pi}\rho_D(\omega)\mathcal{Z}f(x), \tag{11}$$

$$\delta V(x') = \sum_n \delta V_n(x') = \sum_n \hbar\frac{\lambda C_n}{2m_n\omega_n}\mathcal{Z}f(x') = \int_0^\infty\frac{d\omega}{\pi}\rho_D(\omega)\mathcal{Z}f(x'), \tag{12}$$

$$\delta\bar{V}(\bar{x}) = \sum_n \delta\bar{V}_n(\bar{x}) = \sum_n \hbar\frac{\lambda C_n}{2m_n\omega_n}\mathcal{Z}f(\bar{x}) = \int_0^\infty\frac{d\omega}{\pi}\rho_D(\omega)\mathcal{Z}f(\bar{x}), \tag{13}$$

with

$$\mathcal{Z} = \coth\left(\frac{\hbar\omega\beta}{2}\right) = \coth\left(\frac{\hbar\omega}{2k_BT}\right) \tag{14}$$

and

$$\rho_D(\omega) = \sum_n \delta(\omega-\omega_n)\frac{\pi\hbar\lambda C(\omega)}{2m\omega}. \tag{15}$$

Also, in Eq. (10),

$$k^{(2)}(\tau-\sigma) = \frac{M}{\hbar\beta}\sum_{k=-\infty}^{\infty}\zeta_k^{(2)}e^{i\nu_k(\tau-\sigma)}+\int_0^\infty\frac{d\omega}{\pi}I(\omega)(\mathcal{Z}^2-1) \tag{16}$$

with

$$\zeta_k^{(2)} = \frac{8}{M}\int_0^\infty\frac{d\omega}{\pi}I(\omega)\mathcal{Z}\frac{\omega}{[4\omega^2+\nu_k^2]}, \tag{17}$$

and

$$I(\omega) = \sum_n \delta(\omega-\omega_n)\frac{\pi\hbar\lambda^2C^2(\omega)}{(2m\omega)^2} \tag{18}$$

is the spectral density of the environment oscillators;

$$K^{*(2)}(s-i\tau) = \frac{M}{\hbar\beta}\sum_{k=-\infty}^{\infty}[g_k(s)+h_k(s)]e^{i\nu_k\tau}+\int_0^\infty\frac{d\omega}{\pi}I(\omega)(\mathcal{Z}^2-1) \tag{19}$$

with

$$g_k(s) = \frac{8}{M}\int_0^\infty\frac{d\omega}{\pi}I(\omega)\mathcal{Z}\frac{\omega}{[4\omega^2+\nu_k^2]}\cos(2\omega s) \tag{20}$$

and

$$h_k(s)=\frac{4}{M}\int_0^\infty \frac{d\omega}{\pi} I(\omega)\mathcal{Z}\,\frac{\nu_k}{[4\omega^2+\nu_k^2]}\sin(2\omega s). \tag{21}$$

The other two functions in Eq. (10) are defined as

$$\eta^{(2)}(s)=\sum_n \lambda^2 \eta_n^{(2)}(s) = -\int_0^\infty \frac{d\omega}{\pi} 2I(\omega)\mathcal{Z}\sin(2\omega s) \tag{22}$$

and

$$\nu^{(2)}(s)=\sum_n \lambda^2 \nu_n^{(2)}(s) = \int_0^\infty \frac{d\omega}{\pi} I(\omega)[(\mathcal{Z}^2-1)+(\mathcal{Z}^2+1)\cos(2\omega s)]. \tag{23}$$

For the case of factorized initial conditions, Eq. (10) reduces satisfactorily to the influence functional obtained by Hu *et al.* [4].

III. THE PROPAGATOR

In the following, we use the influence functional obtained in Sec. II with $f(x)=x$, i.e., with couplings linear in the system coordinate, to get the propagator for the particle in a harmonic potential and an additional anharmonic potential.

A. Harmonic potential

We consider the quantum particle in a harmonic potential,

$$V(x)=V_h(x,t)\equiv \frac{1}{2}M\omega_0^2x^2-xF(t), \tag{24}$$

in Eq. (1), where $F(t)$ is an external time-dependent force acting on the system. As in the paper by Grabert, Schramm, and Ingold [8], we assume that this force does not influence the initial state, i.e.,

$$F(t)=0, \quad t\leqslant 0. \tag{25}$$

Now the propagator J in Eq. (6) can be written as

$$\begin{aligned}J(x_f,x_f',t,x_i,x_i',\overline{x},\overline{x'})=&\frac{1}{Z}\int DxDx'D\overline{x}\,\exp\Bigg\{\frac{i}{\hbar}\int_0^t ds\left[\frac{1}{2}M\dot{x}^2-V(x)\right]-\frac{i}{\hbar}\int_0^t ds\left[\frac{1}{2}M\dot{x}'^2-V(x')\right]\\&-\frac{1}{\hbar}\int_0^{\hbar\beta}d\tau\left[\frac{1}{2}M\dot{\overline{x}}^2+V(\overline{x})\right]-\frac{i}{\hbar}\int_0^t ds\,\delta V(x)+\frac{i}{\hbar}\int_0^t ds\,\delta V(x')-\frac{1}{\hbar}\int_0^{\hbar\beta}d\tau\,\delta\overline{V}(\overline{x})\\&+\frac{1}{2\hbar}\int_0^{\hbar\beta}d\tau\int_0^{\hbar\beta}d\sigma k^{(2)}(\tau-\sigma)\overline{x}(\tau)\overline{x}(\sigma)+\frac{i}{\hbar}\int_0^t ds\int_0^{\hbar\beta}d\tau K^{*(2)}(s-i\tau)\overline{x}(\tau)[x(s)-x'(s)]\\&-\frac{i}{\hbar}\int_0^t ds\int_0^s du[x(s)-x'(s)]\eta^{(2)}(s-u)[x(u)+x'(u)]\\&-\frac{1}{\hbar}\int_0^t ds\int_0^s du[x(s)-x'(s)]\nu^{(2)}(s-u)[x(u)-x'(u)]\Bigg\},\end{aligned} \tag{26}$$

where Z is a normalization constant. All the terms appearing in J have been defined in the preceding section. It can be seen that the functional integrals in Eq. (26) [with $f(x)=x$ and $f(\overline{x})=\overline{x}$] are Gaussian and hence can be worked out using the fact that significant contribution to the effective action comes from the minimal action paths. Proceeding in a manner similar to Grabert, Schramm, and Ingold [8], we get the propagator as

$$J=\frac{1}{Z}\exp\left\{\frac{i}{\hbar}\sum\,(r_f,q_f,t,r_i,q_i,\overline{r},\overline{q})\right\}, \tag{27}$$

where

$$\begin{aligned}\frac{i}{\hbar}\sum\,(r_f,q_f,t,r_i,q_i,\overline{r},\overline{q})=&-(\alpha_1\overline{r}^2+\alpha_2\overline{q}^2)+i\alpha_3(q_fr_f+q_ir_i)+i\alpha_4q_ir_f\\&+i\alpha_5q_fr_i+i\alpha_6q_i\overline{r}-\alpha_7q_i\overline{q}+i\alpha_8q_f\overline{r}-\alpha_9q_f\overline{q}\\&-(\alpha_{10}q_i^2+\alpha_{11}q_iq_f+\alpha_{12}q_f^2)+i\alpha_{13}q_i+i\alpha_{14}q_f,\end{aligned} \tag{28}$$

with

$$q(s)=x(s)-x'(s), \quad r(s)=\frac{x(s)+x'(s)}{2}, \tag{29}$$

$$q(0)=x_i-x_i'=q_i,\quad r(0)=\frac{x_i+x_i'}{2}=r_i, \tag{30}$$

$$q(t)=x_f-x_f'=q_f,\quad r(t)=\frac{x_f+x_f'}{2}=r_f, \tag{31}$$

$$\overline{q}=\overline{x}-\overline{x'},\quad \overline{r}=\frac{\overline{x}+\overline{x'}}{2}. \tag{32}$$

The various coefficients on the right hand side (rhs) of Eq. (28) are

$$\alpha_1=\frac{M}{2\hbar\wedge},\quad \alpha_2=\frac{M}{2\hbar}\Omega^{(2)},\quad \alpha_3=\frac{M}{\hbar}\frac{\dot G_+(t)}{G_+(t)},$$

$$\alpha_4=-\frac{M}{\hbar}\frac{1}{G_+(t)},\quad \alpha_5=\frac{-M}{\hbar}\frac{1}{G_-(t)},\quad \alpha_6=\frac{M}{\hbar}C_1^{(2)+}(t),$$

$$\alpha_7=-\frac{M}{\hbar}C_2^{(2)+}(t),\quad \alpha_8=\frac{M}{\hbar}C_1^{(2)-}(t),$$

$$\alpha_9=-\frac{M}{\hbar}C_2^{(2)-}(t),$$

$$\alpha_{10}=\frac{M}{2\hbar}R^{(2)++}(t),\quad \alpha_{11}=\frac{M}{\hbar}R^{(2)+-}(t),$$

$$\alpha_{12}=\frac{M}{2\hbar}R^{(2)--}(t),\quad \alpha_{13}=\frac{1}{\hbar}\int_0^t ds\,\frac{G_+(t-s)}{G_+(t)}F_1(s),$$

$$\alpha_{14}=\frac{1}{\hbar}\int_0^t ds\,\frac{G_-(s)}{G_-(t)}F_1(s), \tag{33}$$

with

$$\wedge=\frac{1}{\hbar\beta}\sum_{k=-\infty}^{\infty}u_k^{(2)}, \tag{34}$$

$$u_k^{(2)}=(\nu_k^2+\omega_0^2-\zeta_k^{(2)})^{-1}, \tag{35}$$

$$\nu_k=\frac{2\pi k}{\hbar\beta}, \tag{36}$$

$$\Omega^{(2)}=\frac{1}{\hbar\beta}\sum_{k=-\infty}^{\infty}u_k^{(2)}(\omega_0^2-\zeta_k^{(2)}), \tag{37}$$

$$C_m^{(2)+}(t)=\int_0^t ds\,\frac{G_+(t-s)}{G_+(t)}C_m^{(2)}(s),\quad m=1,2, \tag{38}$$

$$C_m^{(2)-}(t)=\int_0^t ds\,\frac{G_-(s)}{G_-(t)}C_m^{(2)}(s),\quad m=1,2, \tag{39}$$

$$C_1^{(2)}(s)=\frac{1}{\hbar\beta\wedge}\sum_{k=-\infty}^{\infty}u_k^{(2)}g_k(s), \tag{40}$$

$$C_2^{(2)}(s)=\frac{1}{\hbar\beta}\sum_{k=-\infty}^{\infty}\nu_k u_k^{(2)}h_k(s), \tag{41}$$

$$R^{(2)+-}(t)=\int_0^t ds\int_0^t du\,R^{(2)}(s,u)\,\frac{G_+(t-s)}{G_+(t)}\,\frac{G_-(u)}{G_-(t)}, \tag{42}$$

$$R^{(2)}(s,u)=R^{(2)\prime}(s,u)+\frac{\nu^{(2)}(s-u)}{M}, \tag{43}$$

$$R^{(2)\prime}(s,u)=-\wedge C_1^{(2)}(s)C_1^{(2)}(u)+\frac{1}{\hbar\beta}\sum_{k=-\infty}^{\infty}u_k^{(2)}[g_k(s)g_k(u) -h_k(s)h_k(u)], \tag{44}$$

where $G_+(t)$ is a solution of the equation

$$\ddot r+\frac{2}{M}\int_0^s du\,\eta^{(2)}(s-u)r(u)+\omega_0^2 r=0, \tag{45}$$

satisfying $G_+(0)=0$ and $\dot G_+(0)=1$,

$$G_-(s)=\frac{G_+(t-s)\dot G_+(t)-G_+(t)\dot G_+(t-s)}{G_+(t)\ddot G_+(t)-\dot G_+^2(t)} \tag{46}$$

and

$$F_1(s)=\left[F(s)-\int_0^\infty\frac{d\omega}{\pi}\rho_D(\omega)\coth\left(\frac{\hbar\omega}{2k_BT}\right)\right]. \tag{47}$$

The normalization constant Z in Eq. (27) is found to be

$$Z(t)=\left(\frac{2\pi\hbar}{M}\right)^{3/2}\wedge^{1/2}|G_+(t)|. \tag{48}$$

Equations (27)–(48) determine the propagator. It is to be noticed that it has the same form as the corresponding linear coupling case [8] but now with rather complicated coefficients, and the coefficients have an additional temperature-dependent term $\coth(\hbar\omega/2k_BT)$, which appears in the nonlinear coupling problem even for factorized initial conditions [4].

B. Additional anharmonic potential

We now compute the propagator for the quantum particle in a potential

$$V(x)=V_h(x)+V_a(x), \tag{49}$$

where $V_h(x)$ is the harmonic potential (24) and

$$V_a(x)=-V_0\cos(k_0x) \tag{50}$$

is the anharmonic potential. The potential

$$V_w(x)\equiv -V_0\cos(k_0x)-xF \tag{51}$$

is called the washboard potential, with F being a time-*independent* external force. The washboard potential gives an idealized description of the motion of a heavy charged particle in the interior or at the surface of a metal [10], where the underlying crystal provides the periodic potential [the first term on the rhs of Eq. (51)] with lattice constant $2\pi/k_0$, and there is a potential drop $2\pi F/k_0$ per period [the second term on the rhs of Eq. (51)] because of friction with the conduction electrons. The washboard potential is often used to model a single Josephson tunnel junction.

The propagator J as given in Eq. (6) has terms such as

$$\exp\left\{-\frac{i}{\hbar}\int_0^t ds\, V_a(x)\right\},\quad \exp\left\{\frac{i}{\hbar}\int_0^t ds\, V_a(x')\right\},$$

$$\exp\left\{-\frac{1}{\hbar}\int_0^{\hbar\beta} d\tau\, V_a(\overline{x(\tau)})\right\}.$$

Because of the nonlinearities that enter due to $V_a(x)$, a direct evaluation of the functional integrations in J is not possible. However, proceeding in a manner suggested by Fisher and Zwerger [10], we write

$$\exp\left\{-\frac{i}{\hbar}\int_0^t ds\, V_a(x)\right\} = \sum_{n_1=0}^{\infty}\left(\frac{iV_0}{2\hbar}\right)^{n_1}\sum_{\{e_i\}}\int_0^t ds_{n_1}\cdots\int_0^{s_3}ds_2\int_0^{s_2}ds_1 \times\exp\left\{-\frac{i}{\hbar}\int_0^t ds\ \rho(s)x(s)\right\}, \tag{52}$$

where we have introduced n_1 variables or "charges" e_i with a "charge density"

$$\rho(s)=\hbar k_0\sum_{i=1}^{n_1} e_i\delta(s-s_i),\quad e_i=\pm 1. \tag{53}$$

Similarly, expanding the other two terms in J with two new sets of charges σ_k and λ_j, we have

$$\exp\left\{\frac{i}{\hbar}\int_0^t ds\, V_a(x'(s))\right\} = \sum_{n_2=0}^{\infty}\left(-\frac{iV_0}{2\hbar}\right)^{n_2}\sum_{\{\sigma_k\}}\int_0^t ds'_{n_2}\cdots\int_0^{s'_3}ds'_2\int_0^{s'_2}ds'_1 \times\exp\left\{\frac{i}{\hbar}\int_0^t ds\ \rho'(s)x'(s)\right\}, \tag{54}$$

where

$$\rho'(s)=\hbar k_0\sum_{k=1}^{n_2}\sigma_k\delta(s-s'_k),\quad \sigma_k=\pm 1 \tag{55}$$

and

$$\exp\left\{-\frac{1}{\hbar}\int_0^{\hbar\beta} d\tau\, V_a(\overline{x(\tau)})\right\} = \exp\left\{\frac{V_0}{\hbar}\int_0^{\hbar\beta} d\tau\ \cos(k_0\overline{x(\tau)})\right\} = \sum_{n_3=0}^{\infty}\left(\frac{V_0}{2\hbar}\right)^{n_3}\sum_{\{\lambda_j\}}\int_0^{\hbar\beta}d\tau_{n_3}\cdots\int_0^{\tau_3}d\tau_2\int_0^{\tau_2}d\tau_1 \times\exp\left\{-\frac{1}{\hbar}\int_0^{\hbar\beta}d\tau\,\overline{\rho(\tau)}\,\overline{x(\tau)}\right\}, \tag{56}$$

where

$$\overline{\rho(\tau)}=i\hbar k_0\sum_{j=1}^{n_3}\lambda_j\delta(\tau-\tau_j),\quad \lambda_j=\pm 1. \tag{57}$$

We now use Eq. (6) to have the propagator as

$$J(x_f,x'_f,t,x_i,x'_i,\overline{x},\overline{x'}) = \sum_{n_1,n_2,n_3=0}^{\infty}\left(\frac{iV_0}{2\hbar}\right)^{n_1}\left(\frac{-iV_0}{2\hbar}\right)^{n_2}\left(\frac{V_0}{2\hbar}\right)^{n_3} \times\sum_{\{e_i,\sigma_k,\lambda_j\}}\int ds_{n_1}\cdots\int ds_2\int ds_1\int ds'_{n_2}\cdots \times\int ds'_2\int ds'_1\int d\tau_{n_3}\cdots\int d\tau_2 \times\int d\tau_1 J_1(x_f,x'_f,t,x_i,x'_i,\overline{x},\overline{x'};\rho,\rho',\overline{\rho}). \tag{58}$$

Here we assume that the criterion of uniform convergence as established by Chen *et al.* [11] is satisfied in that there exists a V_0, say $\overline{V}$, such that for $V_0<\overline{V}$ the above series has a uniform convergence. In Eq. (58), J_1 contains the functional integrals,

$$J_1(x_f,x_f',t,x_i,x_i',\overline{x},\overline{x'};\rho,\rho',\overline{\rho})=\frac{1}{Z}\int DxDx'D\overline{x}\exp\Bigg\{\frac{i}{\hbar}\int_0^t ds\left[\frac{1}{2}M\dot{x}^2-\frac{1}{2}M\omega_0^2x^2\right]-\frac{i}{\hbar}\int_0^t ds\left[\frac{1}{2}M\dot{x}'^2-\frac{1}{2}M\omega_0^2x'^2\right]$$
$$-\frac{1}{\hbar}\int_0^{\hbar\beta}d\tau\left[\frac{1}{2}M\dot{\overline{x}}^2+\frac{1}{2}M\omega_0^2\overline{x}^2\right]+\frac{i}{\hbar}\int_0^t dsF_1(s)[x(s)-x'(s)]-\frac{i}{\hbar}\int_0^t ds\rho(s)x(s)$$
$$+\frac{i}{\hbar}\int_0^t ds\rho'(s)x'(s)-\frac{1}{\hbar}\int_0^{\hbar\beta}d\tau\overline{\rho(\tau)}\overline{x(\tau)}+\frac{1}{2\hbar}\int_0^{\hbar\beta}d\tau\int_0^{\hbar\beta}d\sigma k^{(2)}(\tau-\sigma)\overline{x(\tau)}\overline{x(\sigma)}$$
$$+\frac{i}{\hbar}\int_0^t ds\int_0^{\hbar\beta}d\tau K^{*(2)}(s-i\tau)\overline{x(\tau)}[x(s)-x'(s)]-\frac{i}{\hbar}\int_0^t ds\int_0^s du[x(s)-x'(s)]$$
$$\times\eta^{(2)}(s-u)[x(u)+x'(u)]-\frac{1}{\hbar}\int_0^t ds\int_0^s du[x(s)-x'(s)]\nu^{(2)}(s-u)[x(u)-x'(u)]\Bigg\}. \tag{59}$$

The advantage of using the representations in Eqs. (52), (54), and (56) is that the functional integrals in Eq. (59) are Gaussian and can be evaluated exactly. Proceeding in the same manner as before, we have

$$J_1(x_f,x_f',t,x_i,x_i',\overline{x},\overline{x'};\rho,\rho',\overline{\rho})=\frac{M}{2\pi\hbar G_+(t)\left(\frac{2\pi\hbar\wedge}{M}\right)^{1/2}}\exp\left\{\frac{\wedge}{2\hbar M}B^2\right\}\exp\{-P\}\exp\Bigg\{-\frac{M}{\hbar}\left[\frac{1}{2\wedge}\overline{r}^2+\frac{\Omega^{(2)}}{2}\overline{q}^2\right]-\frac{iM}{\hbar^2\beta\wedge}\overline{r}\rho_6$$
$$+\frac{iM}{\hbar^2\beta}\overline{q}\rho_7+\frac{iM}{\hbar}(q_fr_f+q_ir_i)\frac{\dot{G}_+(t)}{G_+(t)}-\frac{iM}{\hbar}\left(q_fr_i\frac{1}{G_-(t)}+q_ir_f\frac{1}{G_+(t)}\right)+\frac{iM}{\hbar}\overline{r}(q_iC_1^{(2)+}(t)$$
$$+q_fC_1^{(2)-}(t))+\frac{M}{\hbar}\overline{q}(q_iC_2^{(2)+}(t)+q_fC_2^{(2)-}(t))-\frac{M}{2\hbar}[q_i^2R^{(2)++}(t)+2q_iq_fR^{(2)+-}(t)$$
$$+q_f^2R^{(2)--}(t)]+\frac{i}{\hbar}\int_0^t ds\left[q_i\frac{G_+(t-s)}{G_+(t)}+q_f\frac{G_-(s)}{G_-(t)}\right]\Bigg[F_1(s)-\frac{1}{2}\rho_2(s)+\rho_4(s)$$
$$-i[-\rho_3(s)+\rho_5(s)]\Bigg]+\frac{M}{\hbar}q_i\int_0^t ds\int_0^t duR^{(2)}(s,u)[f_1(u)-f_2(u)]\frac{G_+(t-s)}{G_+(t)}$$
$$+\frac{M}{\hbar}q_f\int_0^t ds\int_0^t duR^{(2)}(s,u)[f_1(u)-f_2(u)]\frac{G_-(s)}{G_-(t)}-\frac{i}{\hbar}r_f\int_0^t ds\rho_1(s)\frac{G_+(s)}{G_+(t)}$$
$$-\frac{i}{\hbar}r_i\int_0^t ds\rho_1(s)\frac{G_-(t-s)}{G_-(t)}-\frac{M}{\hbar}\overline{q}\int_0^t ds[f_1(s)-f_2(s)]C_2^{(2)}(s)$$
$$-\frac{i}{\hbar}\overline{r}\int_0^t ds\int_0^s du\rho_1(s)G_+(s-u)C_1^{(2)}(u)+\frac{i}{\hbar}\overline{r}\int_0^t ds\int_0^t du\rho_1(s)\frac{G_+(s)}{G_+(t)}G_+(t-u)C_1^{(2)}(u)\Bigg\}. \tag{60}$$

Here $q(s)$, $r(s)$, and $\overline{q}$, $\overline{r}$ are as defined in Eqs. (29) and (32), respectively, and

$$\rho_1(s)=\rho(s)-\rho'(s), \tag{61}$$

$$\rho_2(s)=\rho(s)+\rho'(s), \tag{62}$$

$$\rho_3(s)=\frac{k_0}{\hbar\beta^2\wedge}\sum_{k=-\infty}^{\infty}u_k^{(2)^2}\left(\sum_{j=1}^{n_3}\lambda_j\cos(\nu_k\tau_j)\right)g_k(s), \tag{63}$$

$$\rho_4(s)=\frac{k_0}{\beta}\sum_{k=-\infty}^{\infty}u_k^{(2)}\left(\sum_{j=1}^{n_3}\lambda_j\sin(\nu_k\tau_j)\right)h_k(s), \tag{64}$$

$$\rho_5(s)=\frac{k_0}{\beta}\sum_{k=-\infty}^{\infty}u_k^{(2)}\left(\sum_{j=1}^{n_3}\lambda_j\cos(\nu_k\tau_j)\right)g_k(s), \tag{65}$$

$$\rho_6=\frac{\hbar k_0}{M}\sum_{k=-\infty}^{\infty}u_k^{(2)}\left(\sum_{j=1}^{n_3}\lambda_j\cos(\nu_k\tau_j)\right), \tag{66}$$

$$\rho_7=\frac{\hbar k_0}{M}\sum_{k=-\infty}^{\infty}u_k^{(2)}\nu_k\left(\sum_{j=1}^{n_3}\lambda_j\sin(\nu_k\tau_j)\right), \tag{67}$$

$$f_1(s)=\frac{1}{M}\int_0^{t-s}duG_+(t-s-u)\rho_1(u), \tag{68}$$

$$f_2(s)=\frac{1}{M}\frac{G_+(t-s)}{G_+(t)}\int_0^t duG_+(t-u)\rho_1(u), \tag{69}$$

$G_+(t)$ and $G_-(s)$ are as given by Eqs. (45) and (46), respectively. Also,

$$B=\left[\frac{M}{\hbar\beta\wedge}\rho_6+\frac{\dot{G}_+(t)}{G_+(t)}\int_0^t ds\rho_1(s)G_+(s)+C_1^{(2)+}(t)\int_0^t ds\rho_1(s)G_+(s)+\int_0^t ds\rho_1(s)\frac{G_-(t-s)}{G_-(t)}+\int_0^t ds\int_0^s du\rho_1(s)G_+(s-u)C_1^{(2)}(u)-\int_0^t ds\int_0^t du\rho_1(s)\frac{G_+(s)}{G_+(t)}G_+(t-u)C_1^{(2)}(u)\right], \tag{70}$$

$$P=+\frac{1}{\hbar M}\left(\int_0^t ds\rho_1(s)G_+(s)\right)^2\left[-\frac{\Omega^{(2)}}{2}+C_2^{(2)+}(t)-\frac{1}{2}R^{(2)++}(t)\right]-\frac{i}{\hbar^2\beta}\rho_7\int_0^t ds\rho_1(s)G_+(s)-\frac{i}{\hbar M}\left(\int_0^t ds\rho_1(s)G_+(s)\right)\int_0^t du\frac{G_+(t-u)}{G_+(t)}\left[F_1(u)-\frac{1}{2}\rho_2(u)+\rho_4(u)-i[-\rho_3(u)+\rho_5(u)]\right]-\frac{1}{\hbar}\left(\int_0^t ds\rho_1(s)G_+(s)\right)\int_0^t du\int_0^t dvR^{(2)}(u,v)[f_1(v)-f_2(v)]\frac{G_+(t-u)}{G_+(t)}+\frac{1}{\hbar}\left(\int_0^t ds\rho_1(s)G_+(s)\right)\int_0^t du[f_1(u)-f_2(u)]C_2^{(2)}(u). \tag{71}$$

All the other terms are as given before. Equation (58) along with Eqs. (60) to (71) give the required propagator. The extra terms in Eq. (60) compared to Eq. (28) come primarily due to the additional inhomogeneities in the equations of motion arising from the charge densities (53), (55), and (57) for both the imaginary-time paths (because of nonfactorizable initial states) and the real time paths. It can be seen that dropping the anharmonicity we recover the propagator given by Eqs. (27) and (28) in Sec. III A.

This completes the derivation of the propagator for a particle in a harmonic plus an anharmonic potential where the system-environment coupling is nonlinear and initial conditions are quite general. The influence functional is obtained up to the second order of perturbation. The treatment for a quantum particle in the washboard potential was given by Chen *et al.* [11] using generalized initial conditions, but for *linear* system-environment couplings. They assumed an Ohmic spectral density of the reservoir for which the generalized initial state happens to be equivalent to the product (factorized) initial state. Our treatment is more general, in that we do not make any assumptions on the reservoir spectral density, and the various terms are worked out explicitly.

IV. APPLICATIONS

In this section we use propagator (27) obtained in Sec. III A to derive the master equation for the dynamics of the reduced density matrix of the system in a harmonic potential. We first derive the master equation for a general nonfactorizable initial condition and then consider the specific "thermal" initial condition [6]. We also derive the corresponding Wigner equations.

A. General nonfactorizable initial condition

1. The master equation

The time variation of the reduced density matrix of the oscillator is given as

$$\frac{\partial}{\partial t}\rho(q_f,r_f,t)=\int dq_i dr_i d\overline{q}d\overline{r}\frac{\partial}{\partial t}J(q_f,r_f,t,q_i,r_i,\overline{q},\overline{r})\times\lambda_0(q_i,r_i,\overline{q},\overline{r}), \tag{72}$$

where λ_0 is the preparation function describing the deviation of the initial (nonequilibrium) state from the equilibrium distribution. Using Eqs. (27) and (28) for J and the simplified method of Paz [13], we get the master equation as

$$\frac{\partial}{\partial t}\rho(x_f,x_f',t)=i\left[\frac{\hbar}{2M}(\partial_{x_f}^2-\partial_{x_f'}^2)-\frac{\theta(t)}{\hbar}(x_f-x_f')-\frac{\omega^2(t)}{2\hbar}(x_f^2-x_f'^2)\right]\rho(x_f,x_f',t)-\frac{1}{\hbar}\Gamma(t)(x_f-x_f')(\partial_{x_f}-\partial_{x_f'})\rho(x_f,x_f',t)$$
$$-\frac{1}{\hbar^2}D_{pp}(t)(x_f-x_f')^2\rho(x_f,x_f',t)-\frac{i}{\hbar}[D_{xp}(t)+D_{px}(t)](x_f-x_f')(\partial_{x_f}+\partial_{x_f'})\rho(x_f,x_f',t)$$
$$-\frac{1}{\hbar}D_{xx}(t)(\partial_{x_f}+\partial_{x_f'})^2\rho(x_f,x_f',t)+\frac{i}{\hbar}\tilde{C}_1(t)(x_f-x_f')\rho_1(x_f,x_f',t)-\frac{1}{\hbar}\tilde{C}_2(t)(x_f-x_f')\rho_2(x_f,x_f',t). \tag{73}$$

Here we have reverted back to the original coordinates using Eq. (29), and p is the momentum of the particle. The various coefficients in Eq. (73) are

$$\theta(t)=\hbar\left[\frac{\dot{\alpha}_5\alpha_{14}}{\alpha_5}-\dot{\alpha}_{14}\right], \tag{74}$$

$$\omega^2(t)=\hbar\left[\frac{\alpha_3\dot{\alpha}_5}{\alpha_5}-\dot{\alpha}_3\right], \tag{75}$$

$$\Gamma(t)=\frac{\hbar}{2\alpha_5}\left[\frac{\dot{\alpha}_3\alpha_3}{\alpha_4}-\dot{\alpha}_5\right], \tag{76}$$

which is the dissipation term,

$$D_{pp}(t)=\hbar^2\left[\dot{\alpha}_{12}+\frac{2\dot{\alpha}_3\alpha_3\alpha_{12}}{\alpha_4\alpha_5}-\frac{\dot{\alpha}_3\alpha_{11}\alpha_3^2}{\alpha_4^2\alpha_5}+\frac{\dot{\alpha}_5\alpha_{11}\alpha_3}{\alpha_4\alpha_5}-\frac{2\dot{\alpha}_5\alpha_{12}}{\alpha_5}+\frac{\alpha_3^2\dot{\alpha}_{10}}{\alpha_4^2}-\frac{\dot{\alpha}_{11}\alpha_3}{\alpha_4}\right], \tag{77}$$

which causes decoherence in x,

$$D_{xp}(t)+D_{px}(t)=\hbar\left[\frac{2\dot{\alpha}_3\alpha_{12}}{\alpha_4\alpha_5}-\frac{2\dot{\alpha}_3\alpha_3\alpha_{11}}{\alpha_4^2\alpha_5}+\frac{\dot{\alpha}_5\alpha_{11}}{\alpha_4\alpha_5}+\frac{2\dot{\alpha}_{10}\alpha_3}{\alpha_4^2}-\frac{\dot{\alpha}_{11}}{\alpha_4}\right], \tag{78}$$

which is the anomalous diffusion term,

$$D_{xx}(t)=\hbar\left[\frac{\dot{\alpha}_3\alpha_{11}}{\alpha_4^2\alpha_5}-\frac{\dot{\alpha}_{10}}{\alpha_4^2}\right], \tag{79}$$

which generates decoherence in p,

$$\tilde{C}_1(t)=\hbar\left[\dot{\alpha}_8-\frac{\dot{\alpha}_5\alpha_8}{\alpha_5}\right], \tag{80}$$

which causes inhomogeneity in the master equation,

$$\tilde{C}_2(t)=\hbar\left[\dot{\alpha}_9-\frac{\dot{\alpha}_5\alpha_9}{\alpha_5}\right], \tag{81}$$

which causes inhomogeneity in the master equation,

$$\rho_1(x_f,x_f',t)=\int dx_i dx_i' d\overline{q}d\overline{r}\ \overline{r}J(x_f,x_f',t,x_i,x_i',\overline{q},\overline{r})\times\lambda_0(x_i,x_i',\overline{q},\overline{r}), \tag{82}$$

and

$$\rho_2(x_f,x_f',t)=\int dx_i dx_i' d\overline{q}d\overline{r}\ \overline{q}J(x_f,x_f',t,x_i,x_i',\overline{q},\overline{r})\times\lambda_0(x_i,x_i',\overline{q},\overline{r}). \tag{83}$$

The above coefficients of the master equation can be written in a more compact form as follows:

$$\Gamma(t)=-\frac{\hbar}{2}\frac{\dot{W}(t)}{W(t)}, \tag{84}$$

where

$$W(t)=G_+(t)\ddot{G}_+(t)-\dot{G}_+^2(t), \tag{85}$$

$$\omega^2(t)=-\frac{M}{G_+(t)}\left[\frac{2}{\hbar}\dot{G}_+(t)\Gamma(t)+\ddot{G}_+(t)\right]=M\frac{[\dot{G}_+(t)\dddot{G}_+(t)-\ddot{G}_+^2(t)]}{W(t)}, \tag{86}$$

$$\theta(t)=-\frac{2}{\hbar}\Gamma(t)\int_0^t ds\frac{G_-(S)}{G_-(t)}F_1(s)-F_1(t)-\int_0^t ds\left[\ddot{G}_+(t-s)-\frac{G_+(t-s)}{G_+(t)}\ddot{G}_+(t)\right]F_1(s), \tag{87}$$

$$\tilde{C}_1(t)=\hbar\left[\partial_t^2+\frac{2}{\hbar}\Gamma(t)\partial_t+\frac{1}{M}\omega^2(t)\right][G_+(t)\alpha_6(t)], \tag{88}$$

$$\tilde{C}_2(t)=\hbar\left[\partial_t^2+\frac{2}{\hbar}\Gamma(t)\partial_t+\frac{1}{M}\omega^2(t)\right][G_+(t)\alpha_7(t)], \tag{89}$$

$$D_{xx}(t)=0, \tag{90}$$

$$D_{pp}(t)=\hbar M\left[\partial_{t'}^2+\frac{2}{\hbar}\Gamma(t)\partial_{t'}+\frac{1}{M}\omega^2(t)\right]\frac{\partial}{\partial t}U^{(2)}(t,t')|_{t=t'}, \tag{91}$$

$$D_{xp}(t)+D_{px}(t)=\hbar\left[\partial_{t'}^2+\frac{2}{\hbar}\Gamma(t)\partial_{t'}+\frac{1}{M}\omega^2(t)\right]\times U^{(2)}(t,t')|_{t=t'}, \tag{92}$$

with

$$U^{(2)}(t,t')=\int_0^t ds\int_0^{t'}du\;\; G_+(t-s)R^{(2)}(s,u)G_+(t'-u). \tag{93}$$

It can be seen that the above equations have structures similar to those for the linear coupling case [15], though the coefficients here are quite complicated. The last two terms on the rhs of Eq. (73) make the master equation *inhomogeneous*. This implies that for generalized initial conditions, in the case of nonlinear system-environment couplings, it is not possible to obtain an exact Liouville operator L, where L satisfies the equation

$$\frac{\partial\rho}{\partial t}=L\rho. \tag{94}$$

This is a feature of nonfactorizable initial conditions even for the case of linear system-environment couplings [14].

2. The Wigner equation

The Wigner equation is obtained from the master equation by writing [16]

$$\frac{\partial}{\partial t}W(p,x,t)=\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}dy e^{(i/\hbar)py}\left\langle x-\frac{1}{2}y\left|\frac{\partial}{\partial t}\rho\right|x+\frac{1}{2}y\right\rangle. \tag{95}$$

Using Eq. (73) in Eq. (95) we get

$$\begin{aligned}\frac{\partial W}{\partial t}=&-\frac{1}{M}\frac{\partial}{\partial x}pW+\omega^2(t)\frac{\partial}{\partial p}xW+\theta(t)\frac{\partial}{\partial p}W\\&+D_{pp}(t)\frac{\partial^2}{\partial p^2}W-\frac{1}{\hbar}D_{xx}(t)\frac{\partial^2}{\partial x^2}W\\&+[D_{xp}(t)+D_{px}(t)]\frac{\partial^2}{\partial x\partial p}W+\frac{2}{\hbar}\Gamma(t)\frac{\partial}{\partial p}pW\\&-\tilde{C}_1(t)\frac{\partial}{\partial p}W_1-i\tilde{C}_2(t)\frac{\partial}{\partial p}W_2,\end{aligned} \tag{96}$$

where

$$W_1(p,x,t)=\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}dy e^{(i/\hbar)py}\left\langle x-\frac{1}{2}y|\rho_1|x+\frac{1}{2}y\right\rangle \tag{97}$$

and

$$W_2(p,x,t)=\frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}dy e^{(i/\hbar)py}\left\langle x-\frac{1}{2}y|\rho_2|x+\frac{1}{2}y\right\rangle. \tag{98}$$

The Wigner equation may be employed for calculation of various correlation functions in a quasiclassical manner. The equation obtained here for nonlinear couplings and nonfactorizable initial conditions has wider applicability than at obtained earlier by Romero and Paz [15] for the linear coupling case.

B. Thermal initial condition

We now consider the simple case of a thermal initial condition [6], for which the off-diagonal elements of operators in position space of the particle are suppressed in thermal equilibrium.

1. The master equation

The thermal initial condition has the preparation function [14]

$$\lambda_0(q_i,r_i,\overline{q},\overline{r})=f(q_i,r_i)\delta(\overline{q}-q_i)\delta(\overline{r}-r_i) \tag{99}$$

in Eq. (72). Thus, in Eq. (73), we now have

$$\rho_1(q_f,r_f,t)=\int dq_i dr_i r_i J(q_f,r_f,t,q_i,r_i,q_i,r_i)f(q_i,r_i) \tag{100}$$

[cf. Eq. (82)] and

$$\rho_2(q_f,r_f,t)=\int dq_i dr_i q_i J(q_f,r_f,t,q_i,r_i,q_i,r_i)f(q_i,r_i) \tag{101}$$

[cf. Eq. (83)]. Using Eq. (99) the reduced density matrix of the quantum Brownian oscillator becomes

$$\rho(q_f,r_f,t)=\int dq_i dr_i J(q_f,r_f,t,q_i,r_i,q_i,r_i)f(q_i,r_i). \tag{102}$$

Now, from Eq. (27) we have

$$\begin{aligned}J(q_f,&r_f,t,q_i,r_i,q_i,r_i)\\&=\alpha_0\exp\left\{\frac{i}{\hbar}\sum(q_f,r_f,t,q_i,r_i,q_i,r_i)\right\},\end{aligned} \tag{103}$$

where $\alpha_0=1/Z$ with Z given by Eq. (48), and from Eq. (28),

$$\frac{i}{\hbar}\sum (q_f,r_f,t,q_i,r_i,q_i,r_i)$$
$$=-(\alpha_1 r_i^2+\alpha_2 q_i^2)+i\alpha_3(q_f r_f+q_i r_i)+i\alpha_4 q_i r_f$$
$$+i\alpha_5 q_f r_i+i\alpha_6 q_i r_i-\alpha_7 q_i^2+i\alpha_8 q_f r_i-\alpha_9 q_f q_i$$
$$-(\alpha_{10}q_i^2+\alpha_{11}q_i q_f+\alpha_{12}q_f^2)+i\alpha_{13}q_i+i\alpha_{14}q_f. \quad (104)$$

Thus we have

$$\rho_1(q_f,r_f,t)=-\frac{i}{(\alpha_5+\alpha_8)}\partial_{q_f}\rho(q_f,r_f,t)$$
$$-\frac{1}{(\alpha_5+\alpha_8)}\alpha_3 r_f\rho(q_f,r_f,t)$$
$$-\frac{(\alpha_9+\alpha_{11})}{\alpha_4(\alpha_5+\alpha_8)}(\partial_{r_f}-i\alpha_3 q_f)\rho(q_f,r_f,t)$$
$$-\frac{2i\alpha_{12}}{(\alpha_5+\alpha_8)}q_f\rho(q_f,r_f,t)$$
$$-\frac{\alpha_{14}}{(\alpha_5+\alpha_8)}\rho(q_f,r_f,t) \quad (105)$$

and

$$\rho_2(q_f,r_f,t)=\frac{-i}{\alpha_4}(\partial_{r_f}-i\alpha_3 q_f)\rho(q_f,r_f,t). \quad (106)$$

Using Eqs. (105) and (106) in Eq. (73), and with $D_{xx}(t)=0$ [as in Eq. (90)], we obtain the master equation for the case of thermal initial conditions as

$$\frac{\partial}{\partial t}\rho(x_f,x_f',t)=i\left[\frac{\hbar}{2M}(\partial_{x_f}^2-\partial_{x_f'}^2)-\frac{\tilde{\theta}(t)}{\hbar}(x_f-x_f')\right.$$
$$\left.-\frac{\tilde{\omega}^2(t)}{2\hbar}(x_f^2-x_f'^2)\right]\rho(x_f,x_f',t)$$
$$-\frac{1}{\hbar}\tilde{\Gamma}(t)(x_f-x_f')(\partial_{x_f}-\partial_{x_f'})\rho(x_f,x_f',t)$$
$$-\frac{1}{\hbar^2}\tilde{D}_{pp}(t)(x_f-x_f')^2\rho(x_f,x_f',t)$$
$$-\frac{i}{\hbar}[\tilde{D}_{xp}(t)+\tilde{D}_{px}(t)](x_f-x_f')(\partial_{x_f}+\partial_{x_f'})$$
$$\times\rho(x_f,x_f',t). \quad (107)$$

This has the form of an exact master equation, i.e., there are no inhomogeneities, and in this case an exact Liouville operator L exists. This is in agreement with the findings of Karrlein and Grabert [14] for thermal initial conditions in the linear coupling case. The inhomogeneities in the master equation emerge only for the general nonfactorizable initial conditions.

2. The Wigner equation

Proceeding as before, we obtain the Wigner equation from the master equation (107) as

$$\frac{\partial W}{\partial t}=-\frac{1}{M}\frac{\partial}{\partial x}pW+\tilde{\omega}^2(t)\frac{\partial}{\partial p}xW+\tilde{\theta}(t)\frac{\partial}{\partial p}W$$
$$+\tilde{D}_{pp}(t)\frac{\partial^2}{\partial p^2}W+[\tilde{D}_{xp}(t)+\tilde{D}_{px}(t)]\frac{\partial^2}{\partial x\partial p}W$$
$$+\frac{2}{\hbar}\tilde{\Gamma}(t)\frac{\partial}{\partial p}pW. \quad (108)$$

The coefficients on the rhs of Eq. (108) are

$$\tilde{\theta}(t)=\theta(t)+\frac{\alpha_{14}}{(\alpha_5+\alpha_8)}\tilde{C}_1(t), \quad (109)$$

$$\tilde{\omega}^2(t)=\omega^2(t)+\frac{\alpha_3}{(\alpha_5+\alpha_8)}\tilde{C}_1(t), \quad (110)$$

$$\tilde{\Gamma}(t)=\Gamma(t)-\frac{\tilde{C}_1(t)}{2(\alpha_5+\alpha_8)}, \quad (111)$$

$$\tilde{D}_{pp}(t)=D_{pp}(t)+\hbar\left[\frac{\alpha_3(\alpha_9+\alpha_{11})}{\alpha_4(\alpha_5+\alpha_8)}\tilde{C}_1(t)\right.$$
$$\left.-2\frac{\alpha_{12}}{(\alpha_5+\alpha_8)}\tilde{C}_1(t)-\frac{\alpha_3}{\alpha_4}\tilde{C}_2(t)\right], \quad (112)$$

$$\tilde{D}_{xp}(t)+\tilde{D}_{px}(t)=[D_{xp}(t)+D_{px}(t)]$$
$$+\left[\frac{(\alpha_9+\alpha_{11})}{\alpha_4(\alpha_5+\alpha_8)}\tilde{C}_1(t)-\frac{\tilde{C}_2(t)}{\alpha_4}\right]. \quad (113)$$

Equation (108) has the form of a generalized Fokker-Planck equation.

Thus we see that nonlinearity in the environment up to second-order perturbation does *not* introduce any nonlinear behavior in the system for either factorized initial conditions [4] or nonfactorizable initial conditions.

V. FLUCTUATION-DISSIPATION THEOREM

The real and imaginary parts of the coordinate autocorrelation function of the quantum particle are not independent and should be related by a generalized fluctuation-dissipation theorem. In this section we establish a fluctuation-dissipation theorem using the propagator in Eq. (27) for the quantum Brownian particle in a harmonic potential. Proceeding as do Grabert *et al.* [8], we have the response function $\chi(t)$ of the quantum oscillator as

$$\chi(t)=\frac{1}{M}G_+(t) \quad (114)$$

and the coordinate autocorrelation function $C(t)$ as

$$C(t)=\langle x(t)x\rangle = S(t)+iA(t), \tag{115}$$

where $S(t)$ is the symmetrized correlation given by

$$S(t)=\frac{1}{2}\langle x(t)x+xx(t)\rangle = \frac{\hbar}{M}\wedge[\dot{G}_+(t)+G_+(t)C_1^{(2)+}(t)], \tag{116}$$

and $A(t)$ is the antisymmetrized correlation given by

$$A(t)=\frac{1}{2i}\langle x(t)x-xx(t)\rangle = -\frac{\hbar}{2}\chi(t), \quad t\geqslant 0, \tag{117}$$

with $\chi(t)$ given by Eq. (114), these having the same form as in the linear coupling case, and

$$C_1^{(2)+}(t)=\int_0^t ds\frac{G_+(t-s)}{G_+(t)}C_1^{(2)}(s)$$

[Eq. (38) with $m=1$]. Substituting $u_k^{(2)}$ from Eq. (35), $\zeta_k^{(2)}$ from Eq. (17), $\eta^{(2)}(s)$ from Eq. (22) and using

$$\hat{G}_+(z)\equiv\mathcal{L}\{G_+(t)\}=\frac{1}{\left[z^2+\frac{2}{M}\hat{\eta}^{(2)}(z)+\omega_0^2\right]} \tag{118}$$

from Eq. (45), we get

$$u_k^{(2)}=\hat{G}_+(|\nu_k|), \tag{119}$$

$$G_+(t)C_1^{(2)+}(t)=\frac{1}{\hbar\beta\wedge}\sum_{k=-\infty}^{\infty}\hat{G}_+(|\nu_k|)\int_0^t ds G_+(t-s)g_k(s), \tag{120}$$

and

$$\hat{S}(z)\equiv\mathcal{L}\{S(t)\}=\frac{\hbar\wedge}{M}z\hat{G}_+(z) + \frac{1}{\beta M}\sum_{k=-\infty}^{\infty}\hat{G}_+(|\nu_k|)\hat{G}_+(z)\hat{g}_k(z), \tag{121}$$

where $\mathcal{L}$ stands for the Laplace transform. From Eq. (20), we have

$$g_k(s)=\frac{8}{M}\int_0^\infty\frac{d\omega}{\pi}I(\omega)\coth\left(\frac{\hbar\omega}{2k_BT}\right)\frac{\omega}{[4\omega^2+\nu_k^2]}\cos(2\omega s) = \overline{\gamma}^{(2)}(s)-\overline{\zeta}_k^{(2)}(s), \tag{122}$$

where

$$\overline{\gamma}^{(2)}(s)=\frac{2}{M}\int_0^\infty\frac{d\omega}{\pi}\frac{I(\omega)}{\omega}\coth\left(\frac{\hbar\omega}{2k_BT}\right)\cos(2\omega s) \tag{123}$$

and

$$\overline{\zeta}_k^{(2)}(s)=\frac{2}{M}\int_0^\infty\frac{d\omega}{\pi}I(\omega)\coth\left(\frac{\hbar\omega}{2k_BT}\right) \times\frac{\nu_k^2}{\omega[4\omega^2+\nu_k^2]}\cos(2\omega s). \tag{124}$$

Thus, from Eq. (121), we get

$$\hat{S}(z)=\frac{1}{\beta M}\sum_{k=-\infty}^{\infty}\hat{G}_+(|\nu_k|)\hat{G}_+(z)\left\{z+\hat{\overline{\gamma}}^{(2)}(z) - \frac{z}{(z^2-\nu_k^2)}|\nu_k|\hat{\overline{\gamma}}^{(2)}(|\nu_k|)+\frac{\nu_k^2}{(z^2-\nu_k^2)}\hat{\overline{\gamma}}^{(2)}(z)\right\}. \tag{125}$$

Now, using Eq. (118), we have

$$\hat{\eta}^{(2)}(z)=\frac{M}{2\hat{G}_+(z)}-\frac{M}{2}(z^2+\omega_0^2). \tag{126}$$

Also, using Eqs. (22) and (123), we have

$$\hat{\eta}^{(2)}(z)=\frac{M}{2}\{z\hat{\overline{\gamma}}^{(2)}(z)-\overline{\gamma}^{(2)}(0)\}, \tag{127}$$

where

$$\overline{\gamma}^{(2)}(0)=\frac{2}{M}\int_0^\infty\frac{d\omega}{\pi}\frac{I(\omega)}{\omega}\coth\left(\frac{\hbar\omega}{2k_BT}\right). \tag{128}$$

Combining Eqs. (126) and (127), we have

$$\hat{\overline{\gamma}}^{(2)}(z)=\frac{1}{z\hat{G}_+(z)}-\frac{(z^2+\omega_0^2)}{z}+\frac{\overline{\gamma}^{(2)}(0)}{z}. \tag{129}$$

Using Eq. (129) in Eq. (125) we get

$$\hat{S}(z)=\frac{1}{\beta M}\sum_{k=-\infty}^{\infty}\frac{z}{(\nu_k^2-z^2)}[\hat{G}_+(z)-\hat{G}_+(|\nu_k|)]. \tag{130}$$

This can then be cast in the form

$$\tilde{S}(\omega)=\hbar\coth\left(\frac{\hbar\omega}{2k_BT}\right)\tilde{\chi}''(\omega), \tag{131}$$

which is the usual statement of the fluctuation-dissipation theorem, where $\tilde{S}(\omega)$ is the Fourier transform of $S(t)$:

$$\tilde{S}(\omega)=\hat{S}(i\omega)+\hat{S}(-i\omega) \tag{132}$$

and

$$\tilde{\chi}''(\omega)=\frac{i}{2}[\hat{\chi}(i\omega)-\hat{\chi}(-i\omega)], \tag{133}$$

with

$$\tilde{\chi}(\omega)=\int_{-\infty}^{\infty}dt\chi(t)e^{i\omega t}=\hat{\chi}(-i\omega), \quad (134)$$

where $\chi(t)$ is the response function (114).

It is thus seen that for our case of couplings nonlinear in the environment coordinates and treated up to second order of perturbation, the form of the fluctuation-dissipation theorem is preserved for factorized initial conditions [4] as well as general nonfactorizable initial conditions. The proportionality of $\tilde{S}(\omega)$ and $\tilde{\chi}''(\omega)$ illustrates the close connection between fluctuation and dissipation mechanisms acting on the quantum Brownian oscillator. That the fluctuation-dissipation relation in our case is not violated serves as an important check of the correctness of our calculations.

VI. SUMMARY

In this paper we have investigated the quantum Brownian motion (QBM) with couplings nonlinear (quadratic) in the environment coordinates, treating it up to second order of perturbation for general *nonfactorizable* initial conditions. We have thus extended the work of Hu *et al.* [4] and Brun [5] who set out the basic foundations for handling nonlinear QBM with *factorized* initial conditions.

We have constructed the influence functional for nonlinear interactions up to second order of perturbation with generalized initial conditions. We have then used the influence functional, restricting the nonlinearity to the environment, to get the propagator for the particle in a harmonic potential as well as for the particle in an additional anharmonic potential, called the washboard potential used to describe the ideal motion of a heavy charged particle in a metal. For the harmonic potential case, the propagator is similar to at in the corresponding linear coupling case [8] even though the coefficients are now more complicated—among other things having an additional temperature-dependent factor in them. For the case of the particle in the washboard potential, we have been able to work out all the terms in the propagator explicitly. This is a step forward from the previous treatment by Chen *et al.* [11] of the case of linear system-environment couplings and an Ohmic spectrum of the reservoir.

From the propagator for the particle in a harmonic potential we have obtained the master equation and the Wigner equation. Both these equations exhibited inhomogeneities, which imply that it is not possible to construct an exact Liouville operator for generalized initial conditions for either the linear coupling case of Karrlein and Grabert [14], Romero and Paz [15], or when there is nonlinearity in the environment coordinate in the system-environment interaction. We have then considered the specific case of a simple initial condition, called the thermal initial condition, where an exact master equation and a Wigner equation resembling the generalized Fokker-Planck equation are obtained. Thus under such simpler initial conditions, an exact Liouville operator exists for the linear [14] as well as the nonlinear coupling case.

We have also used the propagator for the quantum Brownian particle in a harmonic potential to establish a generalized fluctuation-dissipation theorem. Even though the coefficients in our propagator are more complicated than the corresponding linear coupling case, the form of the fluctuation-dissipation relation is found to be the same as that in the linear coupling case, for both factorized [4] and nonfactorizable initial conditions, confirming that the results are physically consistent, and the same physical mechanism is responsible both for the fluctuations of the position of the quantum oscillator and for its damping.

The results presented here are applicable to all the physical problems modeled by the quantum Brownian motion with initially correlated and nonlinearly coupled environment.

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[1] R.P. Feynman and F.L. Vernon, Ann. Phys. (N.Y.) **24**, 118 (1963).
[2] The equations of motion for open quantum systems may also be derived in a general framework by the projection operator technique, for arbitrary system-environment couplings and very general initial conditions; see, for example, H.-P. Breuer and F. Petruccione, *The Theory of Open Quantum Systems* (Oxford University Press, Oxford, 2002), Chap. 9.
[3] A.O. Caldeira and A.J. Leggett, Physica A **121**, 587 (1983).
[4] B.L. Hu, J.P. Paz, and Y. Zhang, Phys. Rev. D **47**, 1576 (1993).
[5] T.A. Brun, Phys. Rev. D **47**, 3383 (1993).
[6] V. Hakim and V. Ambegaokar, Phys. Rev. A **32**, 423 (1985).
[7] C.M. Smith and A.O. Caldeira, Phys. Rev. A **36**, 3509 (1987); **41**, 3103 (1990).
[8] H. Grabert, P. Schramm, and G.L. Ingold, Phys. Rep. **168**, 115 (1988).
[9] S. Banerjee and R. Ghosh, Phys. Rev. A **62**, 042105 (2000).
[10] M.P.A. Fisher and W. Zwerger, Phys. Rev. B **32**, 6190 (1985).
[11] Y.-C. Chen, J.L. Lebowitz, and C. Liverani, Phys. Rev. B **40**, 4664 (1989).
[12] For details, see S. Banerjee, Ph.D. thesis, Jawaharlal Nehru University, New Delhi, 2003.
[13] J.P. Paz, in *Physical Origin of Time Asymmetry*, edited by J.J. Halliwell *et al.* (Cambridge University Press, Cambridge, 1994).
[14] R. Karrlein and H. Grabert, Phys. Rev. E **55**, 153 (1997).
[15] L.D. Romero and J.P. Paz, Phys. Rev. A **55**, 4070 (1997).
[16] H. Dekker, Phys. Rev. A **16**, 2116 (1977).